

MMP Learning Seminar

Week 86.

Content:

Boundedness of complements for
Fano type varieties

Anti-pluricanonical systems on Fano varieties:

Theorem 1.4: X a d -dimensional Fano type variety.
 (X, B) ε -lc Calabi-Yau, B big, $\text{coeff}(B) \geq \delta > 0$.
Then $|mK_X|$ is birational for some $m := m(d, \varepsilon, \delta)$.

} effective birationality.

Theorem 1.7: Let X be a d -dimensional Fano type variety.
Then X admits a $N(d)$ -complement.

} Boundedness of comp for FT

Theorem 1.11: The class of d -dimensional exceptional Fano varieties forms a bounded family.

} Boundedness of exc Fano.

Definition: X Fano is called exceptional if for every $0 \leq B \sim_{\mathbb{Q}} -K_X$, the pair (X, B) is klt.

Today's goal: Theorem 1.7 (d-1) + Theorem 1.4 (d) + Theorem 1.11 (d)

↙ boundedness of comp ↗ effective birat. ↘ boundedness of exc

⇓
Theorem 1.7 (d)

↖ Goal starting from next week.

Proposition 1 (Lifting of complements):

Let $(X, B+M)$ be a generalized lc pair with $-(K_X+B+M)$ is nef.
 X Fano type. Assume $(X, I'+\alpha M)$ is $\mathbb{Q}$ -factorial generalized plt
for some $I' \geq 0$ and $\alpha \in (0,1)$. Assume $-(K_X+I'+\alpha M)$ ample
and $S = L I' \subseteq L B$. Then:

$$(*) \quad H^0(X, \mathcal{O}_X(-m(K_X+B+M))) \longrightarrow H^0(S, \mathcal{O}_S(-m(K_S+B_S+M_S)))$$

for m so that $m(K_X+B+M)$ is Weil.

Remark: S is Fano type in the previous statement.

Proposition 2 (Lifting of complements).

X Fano type. $(X, B+M)$ is gdl & $K_X + B + M$ nef.

Assume there exists $B_1 \leq B$ & $\alpha \in (0,1)$ s.t

$(X, B_1 + \alpha M)$ is gdl not gklt & $-(K_X + B_1 + \alpha M)$ is big + nef.

Then, there exists a diagram as follows:

$$\begin{array}{ccc} \text{Fano type } S' & \longleftrightarrow & X' \\ & & \downarrow \varphi \\ & & X \end{array} \quad \begin{array}{l} \bullet \varphi \text{ only extracts } S' \\ \bullet X' \text{ Fano type.} \end{array}$$

$$H^0(X', \mathcal{O}_{X'}(-m(K_{X'} + \varphi_*^{-1}B + S' + M))) \twoheadrightarrow H^0(S', \mathcal{O}_{S'}(-m(K_{S'} + B_{S'} + M_{S'})))$$

surjective, whenever $m(K_X + B + M)$ is Weil.

Remark: $(X, B+M)$ generalized dlt & X Fano type

$(X, B+M)$ is not gklt & $\{B\} + M$ is big and nef.

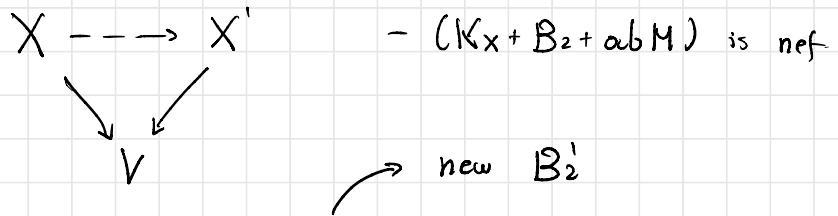
Then we can lift complements from lower dimensions.

Proof:

Step 1: Find new divisor $B_2 \in B_1$.

$B_2 = b B_1$ with $b < 1$. $X \longrightarrow V$ defined $-(K_X + B_1 + a M)$

Run $-(K_X + B_2 + ab M)$ -MMP over V .



$-(K_{X'} + B_2' + ab M)(1-t) - (K_{X'} + B_1' + a M)t$ is big & nef.
for t small enough.

replace B_2 with $(1-t) B_2' + B_1't$
 $ab M$ with $(1-t) ab M + at M$
 X with X'

- i) $(X, B + M)$ gdlit $-(K_X + B + M)$ nef
- ii) $(X, B_1 + a M)$ gdlit not gklt & $-(K_X + B_1 + a M)$ big & nef.
- iii) $(X, B_2 + ab M)$ gklt & $-(K_X + B_2 + ab M)$ big & nef.

Step 2: We produce an anti-ample pair.

$$-(K_X + B_1 + \alpha M) \sim_a A + E$$

$\uparrow$ ample $\uparrow$ effective

Does E contain any glcc of $(X, B_1 + \alpha M)$?

No. $(X, B_1 + \varepsilon E + \alpha M) \rightarrow$ gen dlt.

$$-(K_X + B_1 + \varepsilon E + \alpha M) \sim_a (1-\varepsilon) \left(\underbrace{\frac{\varepsilon}{1-\varepsilon} A}_{\text{ample}} + \underbrace{(-K_X + B_1 + \alpha M)}_{\text{nef}} \right)$$

$\underbrace{\hspace{15em}}_{\text{antiample}}$

$\downarrow$
 gen plt

Yes. $t := \text{glcc}((X, B_2 + \alpha b M), E + B_1 - B_2)$. We have.

$$(X, B_2 + t(E + B_1 - B_2) + \alpha b M) \leftarrow \text{glc.} \quad t > 0$$

$$-(K_X + B_2 + t(E + B_1 - B_2) + \alpha b M) =$$

$$-(K_X + B_1 + \alpha M) + B_1 - B_2 - t(E + B_1 - B_2) + \alpha(1-b)M \sim_a$$

$$A + E + (1-t)(B_1 - B_2) - tE + \alpha(1-b)M =$$

$$tA + (1-t)(A + E + (B_1 - B_2)) + \alpha(1-b)M \sim_a$$

$$tA - (1-t)(K_X + B_2 + \alpha b M) + t\alpha(1-b)M.$$

$\underbrace{\hspace{15em}}_{\text{nef}}$

$\underbrace{\hspace{15em}}_{\text{ample}}$

$$\textcircled{H} = B_2 + t(E + B_1 - B_1)$$

$(X, \mathcal{H} + abM)$ gen lc + anti-ample.

Step 3: Reduce to the gplt case.

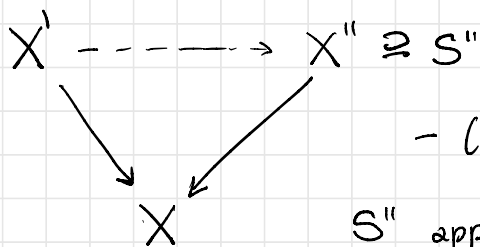
$L(\mathcal{H}) \neq 0$. Then, perturbing coefficients we obtain gplt.

$$L(\mathcal{H}) = 0 \quad (X', \mathcal{H}' + abM) \xrightarrow{\psi} (X, \mathcal{H} + abM)$$

$\mathbb{Q}$ -fact gen dlt modification

E = reduced exceptional of ψ .

$(K_{X'} + E + abM)$ -MMP over X , by neg lemma it terminates on X .



gplt not gplt

$-(K_{X''} + S'' + abM'')$ ample over X

S'' appears with coeff 1 in

$(X'', \mathcal{H}'' + abM'')$ $\rightarrow$ anti-big + anti-nef

gplt not gplt

Taking linear combinations, we get to the case.

gplt + not gplt anti-ample.

□.

Proposition 3 (Complements for non gklt pairs):

Assume existence of bounded complements for Fano type pairs } //

of dimension $d-1$.

$(X, B+M)$ glc & anti-nef. X Fano type.

$(X, B+M)$ is not gklt, either $K_X+B+M \not\sim_{\mathbb{Q}} 0$ or $M \not\sim_{\mathbb{Q}} 0$.

Then $(X, B+M)$ admits a bounded complement.

$$N := -(K_X+B+M)$$

$\uparrow$ b-nef divisor

$$K_X+B+\underbrace{M+N}_{\sim_{\mathbb{Q}} 0} \sim_{\mathbb{Q}} 0$$

Proof:

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longleftarrow & V \end{array}$$

M -trivial.

defined by $-(K_X+B+M)$

M -MMP over $\mathbb{Z}$

• $\dim V < \dim X'$, then we can use the canonical bundle formula and lift complements from V .

• $\dim V = \dim X'$, M is big + nef

$$(X, B+\alpha M) \quad \text{with } \alpha < 1$$

$-(K_X+B+\alpha M)$ is big and nef.

The statement follows by Proposition 2.

Remark: Now, we have bounded complements
for glc pairs which are not gklt & $M \not\sim_{\mathbb{Q}} 0$.

Proposition 4 (Complements for strongly non-exceptional)

Assume existence of bounded complements for Fano type pairs of dimension $d-1$.

X Fano type. $(X, B+M)$ is glc, $-(K_X+B+M)$ nef.

$(X, B+M)$ is strongly non-exc. Then $(X, B+M)$ admits a bounded comp.

Proof: $(X, B+P_{\mathbb{Q}}+M)$ non-glc comp. $t = \text{glc}((X, B+M); P) < 1$.

$(X', \Omega' + M')$ gklt of $(X, B+tP+M)$

↓ perturb the coeff s.t it has the same coeff than B .

$-(K_{X'} + \Omega' + M')$ -MMP until it is semiample

When semiample, we have that $-(K_{X'} + \Omega' + M') \not\sim_{\mathbb{Q}} 0$.

We can apply Prop 3 to conclude that

$(X', \Omega' + M')$ and hence $(X, B+M)$ admits a bounded complement

□

Proposition 5 (Index conjecture for Fano type pairs):

Assume existence of bounded complements for Fano type pairs of dimension $d-1$.

X d -dimensional Fano type, (X, B) lc, $\text{coeff}(B) \leq R$ ↑ finite

$K_X + B \sim_{\mathbb{Q}} 0$. Then $N(K_X + B) \sim 0$ for some N only depending on d & R .

Step 1: We reduce to the case in which X is ε -lc.
& $\rho(X) = 1$

This is a simple application of ACC for lc's.

If a div over X has log discrepancy $\varepsilon > 0$ → only dep on R & d .

we can extract it and increase its coeff to 1.

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \psi \downarrow & \text{MFS} & \\ T & & \end{array}$$

$\dim T > 0$, canonical bundle formula

$$\eta(K_{X'} + B') \sim \eta \psi^*(K_T + B_T + M_T)$$

↑
controlled index
by ind on dimension.

We may assume $T = \text{pt.}$ $\rho(X') = 1$.

Step 2: Introduce some divisors A & R .

By Theorem 1.4: $| -nK_X |$ defines a birational map.

$$\begin{array}{c} X' \\ \phi \downarrow \\ X \end{array}$$

$$\phi^* (-nK_X) \sim A' + R'$$

A will be the pushforward to X of a general member of $|A'|$.

$$R = \phi_* R'$$

$$(X, B) \text{ is lc}$$

$$n \left(K_X + \frac{1}{n} A + \frac{1}{n} R \right) \sim 0$$

$$(X, \frac{1}{n} A + \frac{1}{n} R) \text{ is lc.}$$

Step 3: We define Δ & N .

$$\Delta = \frac{1}{2} B + \frac{1}{2n} R \quad N = \frac{1}{2n} A.$$

→ gen pair with b-nef divisor N

$$\begin{aligned} 2n(K_X + \Delta + N) &= 2n\left(K_X + \frac{1}{2} B + \frac{1}{2n} R + \frac{1}{2n} A\right) \\ &= n(K_X + B) + \boxed{nK_X + R + A} \sim 0. \\ &\sim n(K_X + B) \end{aligned}$$

$K_X + \Delta + N \sim 0 \rightarrow$ not gklt (glc) then we are done.

Step 4: We show the statement when $(X, \Delta + N)$ is klt.

$$\varepsilon' = \min \left\{ \frac{\varepsilon}{2}, \frac{1}{2n} \right\}. \quad \text{Claim: } (X, \Delta + N) \text{ is } \varepsilon'\text{-lc.}$$

$$0 < \alpha(D, X, \Delta + N) < \varepsilon'.$$

$$\alpha(D, X, \Delta + N) = \frac{1}{2} \alpha(D, X, B) + \frac{1}{2} \alpha(D, X, \frac{1}{n} R + \frac{1}{n} A)$$

$\downarrow$ then $> \varepsilon$ $\downarrow$ then $\geq \frac{1}{n}$

$(X, \Delta + N)$ klt, $\Delta + N$ big, coeff $\Delta + N$ are in a finite set }
 $K_X + \Delta + N$ is $\mathbb{Q}$ -trivial $(X, \Delta + N)$ is ε' -lc

Hacon - Xu $\rightarrow$ X belongs to a bounded family. □

Proposition 6 (Complements for non-exceptional)

Assume existence of bounded complements for Fano type pairs of dimension $d-1$.

X Fano type. $(X, B+M)$ is glc, $-(K_X+B+M)$ nef.

$(X, B+M)$ is ~~strongly~~ non-exc. Then $(X, B+M)$ admits a bounded complement.

Proof: $(X, B+P_{\frac{1}{2}}+M)$ non-glc comp. $t = \text{glc}((X, B+M); P) \leq 1$.

$(X', \Omega' + M')$ gdlc of $(X, B+tP+M)$

↓ perturb the coeff s.t it has the same coeff than B .

$-(K_{X'} + \Omega' + M')$ -MMP until it is semiample

When semiample, we have that $-(K_{X'} + \Omega' + M')$

We can apply Prop 3 to conclude that

$(X', \Omega' + M')$ and hence $(X, B+M)$ admits a bounded complement } If χ_{ao}

If

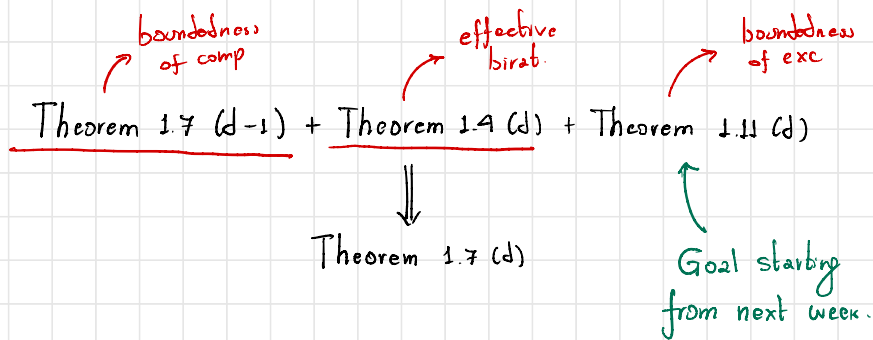
$-(K_{X'} + \Omega' + M') \sim_{\mathbb{Q}} 0$

The only case in which we cannot apply

Prop 3 is when $M \sim_{\mathbb{Q}} 0$.

In this case $K_{X'} + \Omega' \sim_{\mathbb{Q}} 0$ so we can apply Proposition 5.

□



Proof: $(X, B+M)$ as in the statement of Thm 1.7 in dimension d .

If $(X, B+M)$ is non-exc, then Prop 6 implies it admits a bounded complement.

If $(X, B+M)$ is exc. Thm 1.11 (d) implies it belongs to a bounded family. In this bounded family we can find a universal constant for which

$\Gamma \in |-N(K_X + B + M)|$ is a bpf linear system

$\uparrow$
 general

$(X, B + \Gamma/N + M)$ is a N -comp.

□.