MMP Learning Seminar

Week 86.

Content:

Boundedness of complements for Fano type varieties

Anti-pluricanonical systems on Fano varieties:

Theorem 1.4: X a d-dimensional Fano type variety.
(X, B) ε-lc Calabi-Yau, B big, $\mathrm{coeff}(B) \geq \delta > 0$.
Then $|-mK_X|$ is birational for some $m := m(d, \varepsilon, \delta)$.

effective birationality.

Theorem 1.7: Let X be a d-dimensional Fano type variety.
Then X admits a $N(d)$-complement.

Boundedness of comp for FT

Theorem 1.11: The class of d-dimensional exceptional Fano varieties forms a bounded family.

Boundedness of exc Fanos.

Definition: X Fano is called exceptional if for every
$0 \leq B \sim_{\mathbb{Q}} -K_X$, the pair (X, B) is klt.

boundedness of comp

effective birat.

boundedness of exc

Today's goal: $\underline{\text{Theorem 1.7 } (d-1)} + \underline{\text{Theorem 1.4 } (d)} + \text{Theorem 1.11 } (d)$

$$\Downarrow$$

Theorem 1.7 (d)

Goal starting from next week.

Proposition 1 (Lifting of complements):

Let $(X, B+M)$ be a generalized lc pair with $-(K_X+B+M)$ is nef. X Fano type. Assume $(X, \Gamma+\alpha M)$ is $\mathbb{Q}$-factorial generalized plt for some $\Gamma \geq 0$ and $\alpha \in (0,1)$. Assume $-(K_X+\Gamma+\alpha M)$ ample and $S = \lfloor \Gamma \rfloor \subseteq \lfloor B \rfloor$. Then:

$$(*) \quad H^0(X, \mathcal{O}_X(-m(K_X+B+M))) \twoheadrightarrow H^0(S, \mathcal{O}_S(-m(K_S+B_S+M_S))).$$

for m so that $m(K_X+B+M)$ is Weil.

Remark: S is Fano type in the previous statement.

Proposition 2 (Lifting of complements).

X Fano type. $(X, B+M)$ is gdlt & $-(K_X+B+M)$ nef.

Assume there exists $B_1 \leq B$ & $\alpha \in (0,1)$ s.t

$(X, B_1 + \alpha M)$ is gdlt not gklt & $-(K_X + B_1 + \alpha M)$ is big + nef.

Then, there exists a diagram as follows:

$$\begin{tikzcd} \text{Fano type } S' \arrow[r, hook] & X' \arrow[d, "\varphi"] \\ & X \end{tikzcd}$$

- φ only extracts S'
- X' Fano type.

$$H^0(X', \mathcal{O}_{X'}(-m(K_{X'} + \varphi_*^{-1}B + S' + M'))) \twoheadrightarrow H^0(S', \mathcal{O}_{S'}(-m(K_{S'} + B_{S'} + M_{S'})))$$

surjective, whenever $m(K_X + B + M)$ is Weil.

Remark: $(X, B+M)$ generalized dlt & X Fano type

$(X, B+M)$ is not gklt & $\underline{\{B\} + M}$ is big and nef.

Then we can lift complements from lower dimensions.

Proof:

Step 1: Find new divisor $B_2 \leq B_1$.

$B_2 = bB_1$ with $b < 1$. $\quad X \longrightarrow V$ defined $-(K_X + B_1 + aM)$

Run $-(K_X + B_2 + abM)$ – MMP over V.

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & V & \end{tikzcd} \qquad -(K_X + B_2 + abM) \text{ is nef}$$

new B_2'

$-(K_{X'} + B_2' + abM)(1-t) - (K_{X'} + B_1' + aM)t$ is big & nef.
for t small enough.

replace B_2 with $(1-t)B_2' + B_1't$

abM with $(1-t)abM + atM$

X with X'

i) $(X, B+M)$ gdlt $\quad -(K_X + B + M)$ nef

ii) $(X, B_1 + aM)$ gdlt not gklt & $-(K_X + B_1 + aM)$ big & nef.

iii) $(X, B_2 + abM)$ gklt & $-(K_X + B_2 + abM)$ big & nef.

Step 2: We produce an anti-ample pair.

$$-(K_X+B_1+\alpha M) \sim_{\mathbb{Q}} \underset{\text{ample}}{A} + \underset{\text{effective}}{E}$$

Does E contain any glcc of $(X, B_1+\alpha M)$?

No. $(X, B_1+\varepsilon E + a M) \longrightarrow$ gen dlt.

$$\underbrace{-(K_X+B_1+\varepsilon E+aM)}_{\text{gen dlt}} \sim_{\mathbb{Q}} (1-\varepsilon)\underbrace{\left(\underbrace{\frac{\varepsilon}{1-\varepsilon}A}_{\text{ample}} + \underbrace{(-K_X+B_1+aM)}_{\text{nef}}\right)}_{\text{antiample}}$$

$\Downarrow$

gen plt

Yes. $t := \mathrm{glct}((X, B_2+abM); E+B_1-B_2)$. We have

$(X, B_2+t(E+B_1-B_2)+abM) \leftarrow$ glc. $t>0$

$$-(K_X+B_2+t(E+B_1-B_2)+abM) =$$

$$-(K_X+B_1+\alpha M) + B_1-B_2 - t(E+B_1-B_2)+\alpha(1-b)M \sim_{\mathbb{Q}}$$

$$A+\overset{\sim_{\mathbb{Q}}}{E} + (1-t)(B_1-B_2) - tE + a(1-b)M =$$

$$tA + (1-t)(A+E+(B_1-B_2)) + a(1-b)M \sim_{\mathbb{Q}}$$

$$tA - (1-t)(\underbrace{K_X+B_2+abM}_{\text{nef}}) + ta(1-b)M.$$

ample.

H $= B_2+t(E+B_2-B_1)$

$(X, \Theta + abM)$ gen lc + anti-ample.

Step 3: Reduce to the gplt case

$\lfloor \Theta \rfloor \neq 0$. Then, perturbing coefficients we obtain gplt.

$\lfloor \Theta \rfloor = 0$ $\quad (X', \Theta' + abM) \xrightarrow{\psi} (X, \Theta + abM)$

$\mathbb{Q}$-fact gen dlt modification

E = reduced exceptional of ψ.

$(K_{X'} + E + abM)$-MMP over X, by neg lemma it terminates on X.

$$\begin{tikzcd} X' \arrow[rr, dashed] \arrow[dr] & & X'' \supseteq S'' \arrow[dl] \\ & X & \end{tikzcd}$$

gdlt not gklt

$-(K_{X''} + S'' + abM'')$ ample over X

S'' appears with coeff 1 in

$(X'', \Theta'' + abM'')$ $\longrightarrow$ anti-big + anti-nef

gdlt not gklt

Taking linear combinations, we get to the case:

gdlt + not gklt anti-ample. $\square$.

Proposition 3 (Complements for non gklt pairs):

Assume existence of bounded complements for Fano type pairs of dimension $d-1$. } //

$(X, B+M)$ glc & anti-nef. X Fano type

$(X, B+M)$ is not gklt, either $K_X+B+M \not\sim_{\mathbb{Q}} 0$, or $M \not\sim_{\mathbb{Q}} 0$.

Then $(X, B+M)$ admits a bounded complement.

$$N := -(K_X+B+M) \qquad K_X+B+\underbrace{M+N}_{\sim_{\mathbb{Q}} 0} \sim_{\mathbb{Q}} 0$$

$\uparrow$ b-nef divisor

Proof:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & X' \arrow[d, "M\text{-trivial}"] \\ Z & V \arrow[l] \end{tikzcd}$$

defined by $-(K_X+B+M)$

M - MMP over Z

- $\dim V < \dim X'$, then we can use the canonical bundle formula and lift complements from V.

- $\dim V = \dim X'$, M is big + nef

  $(X, B+\alpha M)$ with $\alpha < 1$

  $-(K_X+B+\alpha M)$ is big and nef. The statement follows by Proposition 2.

Remark: Now, we have bounded complements for glc pairs which are not gklt & $M \not\sim_{\mathbb{Q}} 0$.

Proposition 4 (Complements for strongly non-exceptional)

Assume existence of bounded complements for Fano type pairs of dimension $d-1$.

X Fano type. $(X, B+M)$ is glc, $-(K_X+B+M)$ nef.

$(X, B+M)$ is strongly non-exc. Then $(X, B+M)$ admits a bounded comp.

Proof: $(X, B+P_{V'_0}+M)$ non-glc comp. $t = \mathrm{glc}((X,B+M); P) < 1$.

$(X', \Omega'+M')$ gdlt of $(X, B+tP+M)$

$\downarrow$ perturb the coeff s.t it has the same coeff than B.

$-(K_{X'}+\Omega'+M')$ -MMP until it is semiample

When semiample, we have that $-(K_{X'}+\Omega'+M') \not\sim_{\mathbb{Q}} 0$.

We can apply Prop 3 to conclude that

$(X', \Omega'+M')$ and hence $(X, B+M)$ admits a bounded complement

$\square$

Proposition 5 (Index conjecture for Fano type pairs):

Assume existence of bounded complements for Fano type pairs of dimension $d-1$.

finite

X d-dimensional Fano type, (X,B) lc, $\mathrm{coeff}(B) \subseteq \mathcal{R}$

$K_X + B \sim_{\mathbb{Q}} 0$. Then $N(K_X+B) \sim 0$ for some N only depending on d & $\mathcal{R}$.

Step 1: We reduce to the case in which X is ε-lc & $\rho(X)=1$.

This is a simple application of ACC for lct's.

If a div over X has log discrepancy $\varepsilon > 0$ — only dep on $\mathcal{R}$ & d.

we can extract it and increase its coeff to 1.

$$\begin{tikzcd} X \arrow[r, dashed] & X' \arrow[d, "\psi"', "\text{MFS}"] \\ & T \end{tikzcd}$$

$\dim T > 0$, canonical bundle formula

$$q(K_{X'} + B') \sim q\psi^*(K_T + B_T + M_T)$$

controlled index by ind on dimension.

We may assume $T = pt$. $\rho(X') = 1$.

Step 2: Introduce some divisors A & R.

By Theorem 1.4: $|-nK_X|$ defines a birational map.

$$\begin{tikzcd} X' \arrow[d, "\phi"'] \\ X \end{tikzcd}$$

$\phi^*(-nK_X) \sim A' + R'$

A will be the pushforward to X of a general member of $|A'|$.

$R = \phi_* R'$.

(X, B) is lc

$n\left(K_X + \frac{1}{n}A + \frac{1}{n}R\right) \sim 0$ $\qquad$ $\left(X, \frac{1}{n}A + \frac{1}{n}R\right)$ is lc.

Step 3: We define Δ & N.

$$\Delta = \frac{1}{2}B + \frac{1}{2n}R \qquad N = \frac{1}{2n}A.$$

gen pair with b-nef divisor N

$$2n(K_X+\Delta+N) = 2n\left(K_X+\frac{1}{2}B+\frac{1}{2n}R+\frac{1}{2n}A\right)$$

$$= n(K_X+B)+\boxed{nK_X+R+A} \sim 0.$$

$$\sim n(K_X+B)$$

$K_X+\Delta+N \sim_{\mathbb{Q}} 0 \longrightarrow$ not gklt (glc) then we are done.

Step 4 : We show the statement when $(X,\Delta+N)$ is klt.

$\varepsilon' = \min\left\{\frac{\varepsilon}{2}, \frac{1}{2n}\right\}$. Claim: $(X,\Delta+N)$ is ε'-lc.

$0 < a(D, X, \Delta+N) < \varepsilon'$.

$$a(D, X, \Delta+N) = \frac{1}{2}\underbrace{a(D,X,B)}_{\geq 0} + \frac{1}{2}\underbrace{a\left(D,X,\frac{1}{n}R+\frac{1}{n}A\right)}_{\geq 0}$$

then $> \varepsilon$. then $\geq \frac{1}{n}$.

$(X,\Delta+N)$ klt, $\Delta+N$ big, coeff $\Delta+N$ are in a finite set
$K_X+\Delta+N$ is $\mathbb{Q}$-trivial, $(X,\Delta+N)$ is ε'-lc

Hacon – Xu $\longrightarrow$ X belongs to a bounded family. $\square$

Proposition 6 (Complements for non-exceptional)

Assume existence of bounded complements for Fano type pairs of dimension $d-1$.

X Fano type. $(X, B+M)$ is glc, $-(K_X+B+M)$ nef.
$(X, B+M)$ is ~~strongly~~ non-exc. Then $(X, B+M)$ admits a bounded complement.

Proof: $(X, B+P+M)$, $P \geq 0$, non-glc comp. $t = \mathrm{glc}((X, B+M); P) \leq 1$.

$(X', \Omega'+M')$ gdlt of $(X, B+tP+M)$

perturb the coeff s.t it has the same coeff than B.

$-(K_{X'}+\Omega'+M')$-MMP until it is semiample

When semiample, we have that $-(K_{X'}+\Omega'+M')$

We can apply Prop 3 to conclude that $(X', \Omega'+M')$ and hence $(X, B+M)$ admits a bounded complement

If $\not\sim_{\mathbb{Q}} 0$

If $-(K_{X'}+\Omega'+M') \sim_{\mathbb{Q}} 0$

The only case in which we cannot apply Prop 3 is when $M \sim_{\mathbb{Q}} 0$.
In this case $K_{X'}+\Omega' \sim_{\mathbb{Q}} 0$ so we can apply Proposition 5.

$\square$

boundedness of comp

effective birat.

boundedness of exc

Theorem 1.7 (d-1) + Theorem 1.4 (d) + Theorem 1.11 (d)

$\Downarrow$

Theorem 1.7 (d)

Goal starting from next week.

Proof: $(X, B+M)$ as in the statement of Thm 1.7 in dimension d.

If $(X, B+M)$ is non-exc, then Prop 6 implies it admits a bounded complement.

If $(X, B+M)$ is exc. Thm 1.11 (d) implies it belongs to a bounded family. In this bounded family we can find a universal constant for which

$\Gamma \in |-N(K_X+B+M)|$ is a bpf linear system

general

$(X, B+\Gamma/N+M)$ is a N-comp.

$\square$.